Dynamically Complete Models

A model with dependent variable y_t is called dynamically complete, if the set of explanatory variables $\mathbf{x}_t = (x_{t1}, \ldots, x_{tk})$ contains all lags (also of y, if necessary) required for explaining y_t. Formally:

$$\mathbb{E}[y_t|\mathbf{x}_t, y_{t-1}, \mathbf{x}_{t-1}, \ldots] = \mathbb{E}[y_t|\mathbf{x}_t] \tag{1}$$

$\mathbf{x}_t$ is then contemporaneously exogeneous and it can also be shown that the residuals of dynamically complete models are automatically serially uncorrelated, that is, $\mathbb{E}(u_t u_s|x_t, x_s) = 0$ for $s \neq t$, such that the model can be consistently estimated.

Example: Consider the AR(1) model

$$y_t = \beta_0 + \beta_1 y_{t-1} + u_t \tag{2}$$

with

$$\mathbb{E}[u_t|y_{t-1}, y_{t-2}, \ldots] = 0. \tag{3}$$

The model is dynamically complete, since

$$\mathbb{E}[y_t|y_{t-1}, y_{t-2}, \ldots] = \mathbb{E}[y_t|y_{t-1}] = \beta_0 + \beta_1 y_{t-1}. \tag{4}$$

Example (continued).

$x_t = y_{t-1}$ is contemporaneously exogeneous:

(5) $$\mathbb{E}(u_t|x_t) = \mathbb{E}(u_t|y_{t-1}) = 0.$$

However, x_t is not strictly exogeneous:

(6) $$\mathbb{C}\text{ov}(u_t, x_{t+1}) = \mathbb{C}\text{ov}(u_t, y_t) = \mathbb{V}\text{ar}(u_t) > 0.$$

In order to see that the residuals are serially uncorrelated, that is, $\mathbb{E}(u_t u_s|x_t, x_s) = 0$ for $s < t$ (the case $s > t$ follows by symmetry), note first that

(7) $$u_s = y_s - (\beta_0 + \beta_1 y_{s-1})$$

is known at time t, such that by (3):

(8) $$\mathbb{E}(u_t|u_s, x_t, x_s) = \mathbb{E}(u_t|u_s, y_{t-1}, y_{s-1}) = 0.$$

Therefore,

(9) $$\mathbb{E}(u_t u_s|u_s, y_{t-1}, y_{s-1}) = u_s \mathbb{E}(u_t|u_s, y_{t-1}, y_{s-1}) = 0,$$

which holds for all u_s, such that also:

(10) $$\mathbb{E}(u_t u_s|y_{t-1}, y_{s-1}) = \mathbb{E}(u_t u_s|x_t, x_s) = 0,$$

that is, the residuals are serially uncorrelated when controlling for the explanatory variable (and also unconditionally: $\mathbb{E}(u_s u_t) = 0$).